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LINEAR PRESERVERS OF MAJORIZATION ON ℓ^{∞}

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ABSTRACT :

In this paper we give a notation of majorization on ℓ^{∞} and linear Preservers of majorization on closed linear subspace c of Banach space . We extend Df of two functions f_1, f_2 wen D is a bounded linear operator also, we extend some results of [4] abut f, g to several functions

Keywords:

Linear Preservers , majorization, stochastic operators , Banach space.

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INTRODUCTION:

Two vectors $x, y \in \mathbb{R}^n$, the set of all n −tuples of real numbers, $x$is said to be majorized by y, and is denoted by $x \prec y$, whenever $\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}$, $(k = 1, 2, \dots, n-1)$and $\sum_{i=1}^{k} x_i^{\downarrow} = \sum_{i=1}^{k} y_i^{\downarrow}$ Here $x_i^{\downarrow}$denotes the ith largest number between the components of a vector $x \in \mathbb{R}^n$, [5].

It is a well-known fact that for $x, y \in \mathbb{R}^n$, $x \prec y$ if and only if there exists a doubly stochastic $n \times n$ matrix D such that $x = Dy$ (see, for example, [1, 2]). Recall that an $n \times n$ matrix $D = (d_{ij})$ is called doubly stochastic if $d_{ij} \geq 0$, for all $i, j = 1, \ldots, n$, and each of its row sums and column sums are equal to 1.

In finite dimensions, a linear map $T: \mathbb{R}^n \to \mathbb{R}^n$ is said to preserve majorization if whenever $x \prec y$, for $x, y \in \mathbb{R}^n$, then $Tx \prec Ty$. It is known that a linear map $T: \mathbb{R}^n \to \mathbb{R}^n$ preserves majorization if and only if T has one of the following forms.

(i) $T(x) = tr(x)a$, for some $a \in \mathbb{R}^n$.

(ii) $T(x) = \beta P(x) + \gamma tr(x)e$ for some $\beta, \gamma \in \mathbb{R}$ and a permutation

$$P: \mathbb{R}^n \to \mathbb{R}^n$$

In this paper we prove that $\mathrm{D}(\mathrm{f}_1 + \mathrm{f}_2) = \sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} \mathrm{d}_{mn}(\mathrm{f}_1 + \mathrm{f}_2)(\mathrm{n}))\mathrm{e}_m$.

Where $\sum_{n=1}^{\infty} \mathrm{d}_{mn} = 1$, and $\sum_{m=1}^{\infty} \mathrm{d}_{mn} = 1$ $\forall m, n \in \mathbb{N}$, and we prove that following conditions for $\mathrm{f}_r, \mathrm{g}_r \in c$ are equivalent.

(i) $\mathrm{f}_r \prec \mathrm{g}_r$ and $\mathrm{g}_r \prec \mathrm{f}_r$.

(ii) $\mathrm{f}_r = \mathrm{P}\mathrm{g}_r$, for some $\mathrm{P} \in \mathcal{P}$.

Now we discus a Majorization on ℓ^∞ and its closed linear subspace . Let ℓ^∞ be the Banach space of all bounded real sequences [4], with the norm $\forall f \in \ell^\infty, \|f\|_\infty = \sup_{n \in \mathbb{N}} |f(n)|$. Each $f \in \ell^\infty$ can be represented in the form $\sum_{n=1}^{\infty} f(n)_{e_n}$, where the series is understood to be convergent in the weak*-topology. Here $e_n \in \ell^\infty$ denotes the sequence $e_n(j) = 0$ for all $j \neq n$, and $e_n(n) = 1$. Following the same procedure as that of [3], we use doubly stochastic operators [1], on ℓ^∞ to define. The majorization relation on this space. Hence it is necessary first to define these operators on ℓ^∞. We recall that an operator $D_0: \ell^1 \to \ell^1$ is called a doubly stochastic operator on ℓ^1 if it is positive, i.e. $D_0 f \geq 0$ for each non-negative $f \in \ell^1$, and

$$\forall n \in \mathbb{N}, \sum_{m=1}^{\infty} D_0 e_n(m) = 1, \qquad \forall m \in \mathbb{N}, \sum_{n=1}^{\infty} D_0 e_n(m) = 1.$$

The set of all doubly stochastic operators on ℓ^1 is denoted by $\mathcal{DS}(\ell^1)$. We refer to [3, 4], for more details.

Definition .1 A bounded linear operator $D: \ell^\infty \to \ell^\infty$ is called a doubly stochastic operator [1], if there exists a doubly stochastic operator $D_0 \in \mathcal{DS}(\ell^1)$. such that $D = D_0^*$, i.e. for every $f \in \ell^\infty$ and $g \in \ell^1, \langle Df, g \rangle = \langle f, D_0 g \rangle$, where

$\langle\cdot,\cdot\rangle: \ell^\infty \times \ell^1 \to \mathbb{R}$denotes the dual pairing between ℓ^1 and its dual space, ℓ^∞. The set of all doubly stochastic operators on $\ell^\infty$is denoted by $\mathcal{DS}(\ell^\infty)$.

Lemma .2 Let $D \in \mathcal{DS}(\ell^\infty)$. Then there exists a family of non-negative real numbers $\{d_{mn} \mid m,n \in \mathbb{N}\}$ with

$$\forall\, n \in \mathbb{N}, \sum_{n=1}^{\infty} d_{mn} = 1 \quad and \quad \forall m \in \mathbb{N}, \sum_{m=1}^{\infty} d_{mn} = 1 \tag{1}$$

and such that for all $f = \sum_{n=1}^{\infty} f(n)e_n$in ℓ^∞,

$$Df = \sum_{m=1}^{\infty} (\sum_{n=1}^{\infty} d_{mn} f(n))e_m.$$

Proof. Suppose $D_0 \in \mathcal{DS}(\ell^1)$satisfies $D_0^* = D$ and let $d_{mn} := (D_0 e_m)(n)$, for all $m,n \in \mathbb{N}$. Then clearly the family $\{d_{mn} \mid m,n \in \mathbb{N}\}$ satisfies (27). Now for $f = \sum_{n=1}^{\infty} f(n)e_n \in \ell^\infty$and $m \in \mathbb{N}$,

$$\langle Df, e_m\rangle = \langle f, D_0 e_m\rangle = \sum_{n=1}^{\infty} f(n)(D_0 e_m)(n) = \sum_{n=1}^{\infty} d_{mn} f(n).$$

Therefore, $Df = \sum_{m=1}^{\infty}\langle Df, e_m\rangle e_m = \sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} d_{mn} f(n))e_m$.The following lemma which, in some respect, is the converse of the previous lemma, furnishes us with a method to construct doubly stochastic operators on ℓ^∞.

Corollary .3 Let $D \in \mathcal{DS}(\ell^\infty)$. Then there exists a family of non-negative real numbers $\{d_{mn} \mid m,n \in \mathbb{N}\}$ with $\forall \mathrm{m,n} \in \mathbb{N}, \sum_{\mathrm{n}=1}^{\infty} \mathrm{d_{mn}} = 1$, and

$$\sum_{\mathrm{m}=1}^{\infty} \mathrm{d_{mn}} = 1 \quad ,$$

and such that for all $f_1 + f_2 = \sum_{n=1}^{\infty}(f_1 + f_2)(n)e_n$in ℓ^∞,

$$D(f_1 + f_2) = \sum_{m=1}^{\infty} (\sum_{n=1}^{\infty} d_{mn} (f_1 + f_2)(n))e_m.$$

Proof. Suppose $D_0 \in \mathcal{DS}(\ell^1)$satisfies $D_0^* = D$ and let $d_{mn} := (D_0 e_m)(n)$, for all $m,n \in \mathbb{N}$. Then clearly the family $\{d_{mn} \mid m,n \in \mathbb{N}\}$ satisfies (1). Now for $f_1 + f_2 = \sum_{n=1}^{\infty}(f_1 + f_2)(n)e_n \in \ell^\infty$and $m \in \mathbb{N}$,

$$\langle D(f_1 + f_2), e_m\rangle = \langle f_1 + f_2, D_0 e_m\rangle = \sum_{n=1}^{\infty} (f_1 + f_2)(n)(D_0 e_m)(n) = \sum_{n=1}^{\infty} d_{mn} (f_1 + f_2)(n).$$

Therefore,

$$D(f_1 + f_2) = \sum\nolimits_{m=1}^{\infty} \langle D(f_1 + f_2), e_m\rangle e_m = \sum\nolimits_{m=1}^{\infty} \left(\sum\nolimits_{n=1}^{\infty} d_{mn} (f_1 + f_2)(n)\right) e_m .$$

Lemma .4 Let $\{d_{mn} \mid m, n \in \mathbb{N}\}$be a family of non-negative real numbers which satisfies the two relations of (26), in Lemma.2 . Then there exists a doubly stochastic operator $D: \ell^\infty \to \ell^\infty$which is represented by the infinite matrix(d_{mn}) [1], in the sense that

$$\forall f \in \ell^\infty, \forall m \in \mathbb{N}, Df(m) = \sum_{n=1}^{\infty} d_{mn} f(n).$$

Proof. According to [3], Proposition 2.6, there exists a doubly stochastic operator $D_0: \ell^1 \to \ell^1$such that, for all $m, n \in \mathbb{N}, D_0 e_m(n) = d_{mn}$. Let
$D: D_0^* \in \mathcal{DS}(\ell^\infty)$.Then, for all $f \in \ell^\infty$ and all $m \in \mathbb{N}$,

$$\langle Df, e_m \rangle = \langle f, D_0 e_m \rangle = \sum_{n=1}^{\infty} d_{mn} f(n),$$

which proves our claim. According to Lemmas.2 and.4, it is worth noting that, unlike general linear operators on ℓ^∞, a doubly stochastic operator on this space is completely determined by its action on the set $\{e_n | n \in \mathbb{N}\}$. We are now ready to define the majorization relation on ℓ^∞.

Definition.5 For f and g in ℓ^∞, f is said to be majorized by g(or, g majorizes f), and is denoted by $f \prec g$. [4], if there exists $D \in \mathcal{DS}(\ell^\infty)$ for which
$f = Dg$. For a one-to-one map σ: $\mathbb{N} \to \mathbb{N}$, let Pσ: $\ell^\infty \to \ell^\infty$ be defined for each
$f \in \ell^\infty$by

$$P_\sigma f = \sum_{n=1}^{\infty} f(n) e_{\sigma(n)}.$$

Then $P_\sigma$is a well-defined bounded linear operator on ℓ^∞. If, moreover, σ is onto then $P_\sigma$is called a permutation. The set of all permutations on ℓ^∞ is denoted by P. Note that each permutation $P_\sigma \in P$is invertible with $P_\sigma^{-1} = P_{\sigma^{-1}}$. Clearly, every permutation is a doubly stochastic operator. Therefore, if P is a permutation on ℓ^∞ then for each $f \in \ell^\infty, Pf \prec f$. In order to construct other examples for majorization on ℓ^∞, we use the following notation. Let $n \in \mathbb{N}$ and suppose $f_0: \{1, \dots, n\} \to \mathbb{R}$ is an element of $\mathbb{R}^n$. Then for each $f \in \ell^\infty$, we use (f_0, f)
to denote a sequence in ℓ^∞ which is defined as follows.

$$\forall j \in \mathbb{N}, (f_0, f)(j) = \begin{cases} f_0(j) & if \quad j \leq n, \\ f(j-n) \; if & f > n. \end{cases}$$

Theorem .6 For f and g in ℓ^∞, suppose $f \prec g$.

Then inf $g \leq inf\, f \leq sup\, f \leq sup\, g$ and lim $inf\, g(n) \leq$lim $inf\, f(n) \leq$lim $sup\, f(n) \leq$lim $sup\, g(n)$.

PROOF. Let g be non-zero and suppose $D: \ell^{\infty} \to \ell^{\infty}$ is a doubly stochastic operator which satisfies $f = Dg$. The first set of inequalities are clear. To prove the second inequalities, we first note that $f \prec g$ if and only if $f + a \prec g + a$, for each $a \in \mathbb{R}$ considered as a constant sequence. Hence, using a translation, if necessary, we may assume that lim $inf\, g(n) \leq 0 \leq$ lim $sup\, g(n)$. Let $\alpha :=$ lim $sup\, g(n)$. For $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $g(n) < \alpha + \frac{\epsilon}{2}$, for all $n \geq \mathbb{N}$. Let $\{d_{ij} \mid i,j \in \mathbb{N}\}$ be the family of non-negative real numbers corresponding to D, introduced in Lemma(4.2.2).Then there exists $M \in \mathbb{N}$ such that for all

$m \geq M, \sum_{j=1}^{N} d_{mj} < \frac{\epsilon}{2\|g\|_{\infty}}$. Therefore, for any $m \geq M$,

$$f(m) = \sum_{j=1}^{\infty} d_{mj}\, g(j) = \sum_{j=1}^{N} d_{mj}\, g(j) + \sum_{j=N+1}^{N} d_{mj}\, g(j)$$

$$\leq \sum_{j=1}^{N} d_{mj}\, \|g\|_{\infty} + \sum_{j=N+1}^{\infty} d_{mj} \left(\alpha + \frac{\epsilon}{2}\right) < \alpha + \epsilon.$$

Hence lim $sup\, f(n) \leq$ lim $sup\, g(n)$.

The inequalitylim $inf\, g(n) \leq$ lim $inf\, f(n)$ follows easily from the previous argument and the fact that $-f = D(-g)$. We continue this section by considering the majorization relation on these closed subspaces. Let e denote the constant sequence . Then the sets $\{e_N | n \in \mathbb{N}\}$ and $\{e_N | n \in \mathbb{N}\} \cup \{E\}$form, respectively, Schauder bases for c_0 and c. For $f \in c$, we use the notation lim$f$in place of $\lim_{n\to\infty} f(n)$. Then every $f \in c$ has the representation

$f = (\lim f)e + \sum_{n=1}^{\infty}(f(n) - \lim f)e_n$, where the series converges in the norm topology. The next lemma follows directly from Theorem.7 .

Lemma .7 For $f, g \in c, if\ f \prec g$ then $\lim f = \lim g$.

there are sequences $f, g \in \ell^{\infty}$ with $f \prec g$ and

$g \prec f$ without, necessarily, each being a permutation of the other. However, in the spaces c and c_{∞} this does not happen. To see this fact, we need the following lemma whose proof is, in some respect, similar to Theorem 3.5 of [3]. However, for the sake of completeness, we bring here its proof. Let us first introduce some notations. For a real number a,let $\phi_a, \psi_a : \mathbb{R} \to \mathbb{R}$be the non-negative convex functions defined, for each $x \in \mathbb{R}$, by

$$\phi_a(x) = max\{x - a, 0\}, \qquad \psi_a(x) = -min\{a - x, 0\}.$$

Then, for each $f \in c_0$ and all $a > 0$ and $b < 0$, we have

$$\sum_{n\in\mathbb{N}} \phi_a\big(f(n)\big) = \sum_{n\in\mathbb{N}} \phi_a\big(f^+(n)\big), \quad \sum_{n\in\mathbb{N}} \psi_a\big(f(n)\big) = \sum_{n\in\mathbb{N}} \phi_{|b|}\big(f^-(n)\big),$$

where $f^+ = max\{f, 0\}$and $f^- = -min\{f, 0\}$. We recall that for a function $f: \mathbb{N} \to \mathbb{R}$, the support of f ,denoted by $supp(f)$, is the set $\{n \in \mathbb{N} | f(n) \neq 0\}$.

For a non-negative $f \in c_0$, let $\{A_n(f) | n \in \mathbb{N}\}$ be a family of subsets of $supp(f)$ defined, inductively, as follows:

$$A_1(f) = \{k \in supp(f) |\, f(k) = \|f\|_\infty\},$$

and for each $n \geq 2$,

$$A_n(f) = \left\{ k \in supp(f) \,\middle|\, f(k) = \left\| f - \sum_{j \in \cup_{i=1}^{n-1} A_i(f)} f(j) e_j \right\|_\infty \right\}.$$

Clearly $A_n(f) \cap A_m(f) = \emptyset$, for $n = m$, and $supp(f) = \cup_{n\in\mathbb{N}} A_n(f)$. Let f_n denote the value of f on the set $A_n(f)$, if this set is non-empty, and define it equal to 0, if $A_n(f) = \emptyset$. If $A_n(f) = \emptyset$, for some $n \in N$, then $f_1 > f_2 > \cdots > f_n$. If $A_n(f) = \emptyset$then $A_m(f) = \emptyset$, for all $m \geq n$.

Again, for a non-negative $f \in c_0$, let $f_\downarrow$ denote the rearrangement of f in the decreasing order. Therefore there exists a permutation $P_\sigma \in P$ for which $f_\downarrow = P_\sigma f$ and in such a way that $f_\downarrow(n) \geq f_\downarrow(n+1)$, for each $n \in \mathbb{N}$. Clearly $supp(f)$ and $sup(f_\downarrow)$ are in one-to-one correspondence. The same is true for the sets $A_n(f)$ and $\mathrm{A_n(f_\downarrow)}$, for all $n \in \mathbb{N}$. For each $a > 0$ we also have,

$$\sum_{n\in\mathbb{N}} \phi_a\big(f_\downarrow(n)\big) = \sum_{n\in\mathbb{N}} \phi_a\left(f\big(\sigma^{-1}(n)\big)\right) = \sum_{m\in\mathbb{N}} \phi_a\big(f(m)\big).$$

Lemma .8 For $f, g \in c_0$, if $f \prec g$ and

$$\begin{aligned} \forall a > 0, \quad & \textstyle\sum_{n\in\mathbb{N}} \phi_a\big(f(n)\big) = \sum_{n\in\mathbb{N}} \phi_a\big(g(n)\big), \\ \forall a > 0, \quad & \textstyle\sum_{n\in\mathbb{N}} \psi_a\big(f(n)\big) = \sum_{n\in\mathbb{N}} \psi_a\big(g(n)\big), \end{aligned} \tag{2}$$

then there exists a permutation $P \in \mathcal{P}$ such that $f = Pg$.

Proof. We may assume that g is non-zero. By the first equation of (2), for each $a > 0$ we have

$$\sum_{n\in\mathbb{N}} \phi_a\big(f_\downarrow^+(n)\big) = \sum_{n\in\mathbb{N}} \phi_a\big(f^+(n)\big) = \sum_{n\in\mathbb{N}} \phi_a\big(g^+(n)\big) = \sum_{n\in\mathbb{N}} \phi_a\big(g_\downarrow^+(n)\big)$$

Since this is true for each $a > 0$, it is easily seen that $A_n(f_\downarrow^+) = A_n(g_\downarrow^+)$. Therefore, for each $n \in \mathbb{N}$, there is a one-to-one correspondence θ_n between the sets $A_n(f^+)$ and $A_n(g^+)$, from which it follows that there is also a bijection

$\theta^+: supp(g^+) \to supp(f^+)$ which maps $A_n(g^+)$ to $A_n(f^+)$, for each $n \in \mathbb{N}$ with $A_n(f^+) \neq \emptyset$.

Let $D: c \to c$ be a doubly stochastic operator with $f = Dg$.We first show that

$$\forall m \in supp(f^+), \qquad \sum_{n \in supp\,(g^+)} De_n(m) = 1, \tag{3}$$

and

$$\forall m \in supp(g^+), \qquad \sum_{n \in supp\,(f^+)} De_n(m) = 1, \tag{4}$$

First suppose $m \in A_1(f^+)$. If $\lambda := \sum_{n \in A_1(g^+)} De_n(m) < 1$, then

$$0 < f_1 = f(m) = \sum_{n=1}^{\infty} De_n(m)g(n) = \sum_{n \in A_1(g^+)} De_n(m)g_1 + \sum_{n \notin A_1(g^+)} De_n(m)g(n)$$

$\leq \lambda g_1 + (1-\lambda)g_2 < g_1$.

This contradicts the fact that $f_1 = g_1$. Hence $\sum_{n \in A_1(g^+)} De_n(m) = 1$ and therefore $\sum_{n \in supp\,(g^+)} De_n(m) = 1$. Furthermore, by the equations

$$|A_1(g^+)| = |A_1(f^+)| = \sum_{m \in A_1(f^+)} \sum_{n \in A_1(g^+)} De_n(m) = \sum_{n \in A_1(g^+)} \sum_{m \in A_1(f^+)} De_n(m),$$

Where for a set A, $|A|$ denotes its cardinal number, we have also $\sum_{m \in A_1(g^+)} De_n(m) = 1$, for each $n \in A_1(g^+)$, whence $De_n(m) = 0$, for each $m \notin A_1(f^+)$and for all $n \in A_1(g^+)$.

Using induction, a similar argument shows that, for each $k \in \mathbb{N}$with $A_k(f^+) = \emptyset$, we have

$$\forall m \in A_k(f^+), \qquad \sum_{n \in A_k(g^+)} De_n(m) = 1,$$

$$\forall m \in A_k(g^+), \qquad \sum_{m \in A_k(f^+)} De_n(m) = 1.$$

This proves (3) and (4). The second equation of (2) and similar arguments yield a bijection $\theta^-: supp(g^-) \to supp(f^-)$ which maps $A_n(g^-)$ to $A_n(f^-)$, for all $n \in \mathbb{N}$ with non-empty $A_n(f^-)$. We also have the following relations.

$$\forall m \in supp(f^-), \sum_{n \in supp\,(g^-)} De_n(m) = 1, \tag{5}$$

$$\forall m \in supp(g^-), \quad \sum_{n \in supp\,(f^-)} De_n(m) = 1. \tag{6}$$

For a sequence $f \in c_0$, if $N(f) := \mathbb{N} \backslash supp(f)$ then (3), (4), (5), and (6) imply that

$$\forall m \in N(f), \qquad \forall n \notin N(g), De_n(m) = 0,$$

$$\forall m \notin N(f), \qquad \forall n \in N(g), De_n(m) = 0.$$

This shows that

$$\sum_{m \in N(f)} 1 = \sum_{m \in N(f)} \sum_{n \in N(f)} De_n(m) = \sum_{n \in N(g)} \sum_{m \in N(f)} De_n(m) = \sum_{n \in N(g)} 1.$$

Thus $|N(f)| = |N(g)|$. Hence there exists a bijection $\theta^0 : N(g) \to N(f)$. Nowwe can define a bijection $\theta : \mathbb{N} \to \mathbb{N}$ by

$$\forall n \in \mathbb{N}, \qquad \theta(n) = \begin{cases} \theta^+(n) \; n \in supp(g^+), \\ \theta^0(n) \; n \in N(g), \\ \theta^-(n) \; n \in supp(g^-). \end{cases}$$

Let $P = P_\theta$ be the corresponding permutation on c. Then, for each $m \in \mathbb{N}$,

$$Pg(m) = \left(\sum_{n=1}^{\infty} g(n) e_{\theta(n)} \right)(m) = g\big(\theta^{-1}(m)\big).$$

If $m \in supp(f^+)$, then $m \in A_k(f^+)$, for some $k \in \mathbb{N}$ and $\theta^{-1}(m) \in A_k(g^+)$. Hence $g(\theta^{-1}(m)) = g_k = f_k = f(m)$. Thus we have $f(m) = Pg(m)$, for each $m \in supp(f^+)$. Similar arguments are true for $m \in N(f)$ and $m \in supp(f^-)$. There fore $f = Pg$.

Theorem .9 The following conditions for $f, g \in c$are equivalent.

(i)$f \prec g$ and $g \prec f$.

(ii) $f = Pg$, for some $P \in \mathcal{P}$.

Proof. (i)⇒(ii) First assume that f and g are in c_0. Let $D, D' \in \mathcal{DS}$ satisfy $f = Dg$ and $= D'f$. Since for each $a \in \mathbb{R}$, the function $\phi_a$is convex, using Jensen's inequality, we obtain that

$$\phi_a\big(f(n)\big) \le \sum_{m \in \mathbb{N}} De_m(n) \phi_a\big(g(m)\big),$$

for each $n \in \mathbb{N}$. Specially, for $a > 0$ we will have

$$\sum_{n \in \mathbb{N}} \phi_a\big(f(n)\big) \le \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} De_m(n) \phi_a\big(g(m)\big) = \sum_{m \in \mathbb{N}} \sum_{n \in \mathbb{N}} De_m(n) \phi_a\big(g(m)\big)$$

$$= \sum_{m \in \mathbb{N}} \phi_a\big(g(m)\big).$$

Similarly,

$$\sum_{m\in\mathbb{N}} \phi_a\big(g(m)\big) \leq \sum_{n\in\mathbb{N}} \phi_a\big(g(n)\big).$$

Hence $\sum_{m\in\mathbb{N}} \phi_a\big(g(m)\big) = \sum_{n\in\mathbb{N}} \phi_a\big(g(n)\big)$. A similar argument shows that $\sum_{m\in\mathbb{N}} \psi_a\big(g(m)\big) = \sum_{n\in\mathbb{N}} \psi_a\big(g(n)\big)$, for each $a < 0$.Thus Lemma.8implies that there is a permutation P for which $f = Pg$. Now for the general case of

$f, g \in c$, if $f \prec g$ and $g \prec f$ then $\lim f = \lim g$ and $f - (\lim f)e \prec g - (\lim g)e$ and $g - (\lim g)e \prec f - (\lim f)e$. By the previous argument, there is a permutation P such that $f - (\lim f)e = P(g - (\lim g)e)$, whence $f = Pg$.

(ii)⇒(i) Clear. For $f, g \in \ell^\infty$, we use the notation $f \sim g$ whenever$f \prec g$ and $g \prec f$. According to the previous theorem, for $f, g \in c, f \sim g$ if and only if

$f = Pg$ for some permutation $P \in \mathcal{P}$.

Corollary .10 The following conditions for $f_r , g_r \in c$are equivalent.

(i) $f_r \prec g_r$ and $g_r \prec f_r$.

(ii) $f_r = Pg_r$, for some $P \in \mathcal{P}$.

Proof. (i)⇒(ii) First assume that f_r and g are in c_0. Let

$$D_1 + D_2, (D_1 + D_2)' \in (D_1 + D_2)\mathcal{S}$$

satisfy $f_r = (D_1 + D_2)g_r$ and $= (D_1 + D_2)' \mathrm{f}_r$. Since for each $a \in \mathbb{R}$, the function $\phi_a$is convex, using Jensen's inequality, we obtain that

$$\phi_a\big(f_r(n)\big) \leq \sum_{m\in\mathbb{N}} (D_1 + D_2)e_m\,(n)\phi_a\big(g_r(m)\big),$$

for each $n \in \mathbb{N}$. Specially, for $a > 0$ we will have

$$\sum_{n\in\mathbb{N}} \phi_a\big(f_r(n)\big) \leq \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} (D_1 + D_2)e_m\,(n)\phi_a\big(g_r(m)\big)$$
$$= \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}} (D_1 + D_2)e_m\,(n)\phi_a\big(g_r(m)\big) = \sum_{m\in\mathbb{N}} \phi_a\big(g_r(m)\big).$$

Similarly,

$$\sum_{m\in\mathbb{N}} \phi_a\big(g_r(m)\big) \leq \sum_{n\in\mathbb{N}} \phi_a\big(g_r(n)\big).$$

Hence $\sum_{m\in\mathbb{N}} \phi_a\big(g_r(m)\big) = \sum_{n\in\mathbb{N}} \phi_a\big(g_r(n)\big)$. A similar argument shows that $\sum_{m\in\mathbb{N}} \psi_a\big(g_r(m)\big) = \sum_{n\in\mathbb{N}} \psi_a\big(g_r(n)\big)$, for each $a < 0$.Thus Lemma .8implies that there is a permutation P for which $f_r = Pg_r$. Now for the general case of $f_r, g_r \in c$, if $f_r \prec g_r$ and $g_r \prec f_r$ then $\lim f_r = \lim g_r$ and $f_r - (\lim f_r)e \prec g_r - (\lim g_r)e$ and $g_r - (\lim g_r)e \prec$

$f_r - (\lim f_r)e$. By the previous argument, there is a permutation P such that $f_r - (\lim f_r)e = P(g_r - (\lim g_r)e)$, whence
$f_r = Pg_r$.
(ii)⇒(i) Clear. For $f_r, g_r \in \ell^\infty$, we use the notation $f_r \sim g_r$ whenever $f_r \prec g_r$ and $g_r \prec f_r$. According to the previous theorem, for $f_r, g_r \in c, f_r \sim g_r$ if and only if $f_r = Pg_r$ for some permutation $P \in \mathcal{P}$.

In this part of this paper we obtain a characterization of linear preservers of the majorization relation on c. As we will see, the restriction of a linear preserver of majorization to the linear subspace c_0 of c is a majorization preserver on this subspace. Therefore, in order to characterize the structure of these maps on c, we first obtain the same characterization on c_0. Finally, using this result, we determine the structure of these maps on c , [2].

Definition .11 A bounded linear map $T : \ell^\infty \to \ell^\infty$ is called a majorization preserver on ℓ^∞ if for each $f, g \in \ell^\infty, f \prec g$ implies that $Tf \prec Tg$, [4]. We denote the set of all linear majorization preservers $T : \ell^\infty \to \ell^\infty$ by $\mathcal{M}_{\mathrm{Pr}}(\ell^\infty)$. The set of all linear majorization preservers on c and c_0 are denoted, respectively, by $\mathcal{M}_{\mathrm{Pr}}(\mathrm{c})$ and $\mathcal{M}_{\mathrm{Pr}}(\mathrm{c}_0)$. For brevity, in what follows, we use the word preserver instead of majorization preserver.

Example .12 For any $h \in c$, let $T = Th$ be the bounded linear operator on c, defined by $Tf = (\lim f)h$. Then $f \prec g$, in c, implies that $Tf = Tg$. Thus T is a preserver.

For a bounded linear map $T: c \to c$, it is easily seen that for each $m \in N$,

$$\sum_{n=1}^{\infty} |T_{e_n}(m)| \leq \|T\|. \tag{7}$$

Theorem .13 For each $T \in \mathcal{M}_{Pr}(c)$the following statements hold.
(i) $T(c_0) \subseteq c_0$, and therefore $T|_{c_0} \in \mathcal{M}_{Pr}(c_0)$.
(ii) If $lim\, Te = \alpha$, then $\lim Tf = \alpha \lim f$, for each $f \in c$.

Proof. (i) Let $T \in \mathcal{M}_{Pr}(c)$be non-zero. It suffices to show that $Te_n \in c_0$, for all $n \in \mathbb{N}$. Suppose, on the contrary, there exists $n_0 \in \mathbb{N}$.with
$l := \lim Te_{n_0} \neq 0$. Then, since $e_n \prec e_{n_0}$, by Lemma .7, $\lim Te_n = l$, for each $n \in \mathbb{N}$. We first choose $N \in \mathbb{N}$ with $N > \frac{2\|T\|}{|l|}$, and then $m_0 \in \mathbb{N}$ such that $|Te_n(m_0)| > \frac{|l|}{2}$, for each $n = 1, \ldots, N$. Now, using (7), we obtain the following contradiction.

$$\|T\| \geq \sum_{n=1}^{\infty} |Te_n(m_0)| \geq \sum_{n=1}^{N} |Te_n(m_0)| \geq N\frac{|l|}{2} > \frac{2|T|}{|l|} \cdot \frac{|l|}{2} = \|T\|.$$

(ii) For $f \in c$, using the previous part, $T(f - (\lim f)e) \in c_0$. Therefore, $\lim Tf = \lim T((\lim f)e) + \lim T(f - (\lim f)e) = (\lim f) \lim Te$. According to the previous theorem, if $T: c \to c$ is a linear preserver then the restriction of T to the closed subspace c_0 of c is an operator on this subspace, and therefore a linear preserver on c_0. Hence we first obtain the structure of an operator $T \in \mathcal{M}_{Pr}(c_0)$.. To this end, we need the following two lemmas.

Lemma .14 Let $T \in \mathcal{M}_{Pr}(c_0)$. Then for any $m \in \mathbb{N}$ there is at most one $n \in \mathbb{N}$ with $Te_n(m) \neq 0$.

Proof. Suppose that, on the contrary, there exists m_0 and two distinct n_1, n_2 in $\mathbb{N}$, for which $a := Te_{n_1}(m_0)$ and $b := Te_{n_2}(m_0)$ are both non-zero. Let $F \subset \mathbb{N}$ be given by

$$F = \{m \in \mathbb{N} | \, Te_{n_1}(m) = a\}.$$

Then $F \neq \emptyset$. Moreover, since $Te_{n_1} \in c$, F is finite. For $n \neq n_1$, and for all $\alpha, \beta \in \mathbb{R} \alpha e_{n_1} + \beta e_{n_2} \sim \alpha e_{n_1} + \beta e_n$.Therefore, $\alpha Te_{n_1} + \beta Te_{n_2} \sim \alpha Te_{n_1} + \beta e_n$ which, by Theorem .9, implies that

$$\alpha a + \beta b = \left(\alpha Te_{n_1} + \beta Te_{n_2}\right)(m_0) \in \left\{\alpha Te_{n_1}(m) + \beta Te_n(m) | m \in \mathbb{N}\right\}.$$

Thus, according to Lemma 4.6 of [3], there exists $m \in \mathbb{N}$ such that $Te_{n_1}(m) = b$ and $Te_n(m) = b$. Note that, by the definition of the set $F, m \in F$. In short, we saw that

$$\forall n \neq n_1, \exists m \in F \text{ such that } Te_{n_1}(m) = a \text{ and } Ten\,(m) = b.$$

Since F is finite, there exists a fixed element $m \in F$ such that $Ten(m) = b$, for infinitely many $m \in \mathbb{N}$. This contradicts the property declared by (7) .

Let $X_i, i \in I$, and Y be non-empty sets. A family of maps $\sum = \{\sigma_i : X_i \to Y | i \in I\}$ is called mutually disjoint if for all distinct pairs $i_1, i_2 \in I$,

$$Im(\sigma_{i_1}) \cap Im(\sigma_{i_2}) = \emptyset,$$

where by $Im(\sigma)$ we mean the image set of a map σ.We recall that for a one-to-one map $\sigma: \mathbb{N} \to \mathbb{N}$, the bounded linear map $P\sigma: c_0 \to c_0$ is defined by $P\sigma e_n = e_{\sigma(n)}$, for each $n \in \mathbb{N}$.

Lemma .15 Let $D \in \mathcal{DS}$. Then, for a mutually disjoint family of one-to-one maps $\sum = \{\sigma_i : \mathbb{N} \to \mathbb{N} | i \in I\}$, there exists a doubly stochastic operator $\widetilde{D} \in \mathcal{DS}$ such that, as linear operators on c_0, $P_\sigma D = \widetilde{D} P_\sigma$, for each $\sigma \in \sum$.

Proof. For $m, n \in \mathbb{N}$, let $\tilde{d}_{mn}$ be defined by

$$\tilde{d}_{mn} = \begin{cases} De_{\sigma^{-1}(n)}\left(\sigma^{-1}(m)\right) & if\ for\ some\ \sigma\in\sum\ m,n\in\sigma(\mathbb{N}), \\ 0 & if\ for\ some\ \sigma\in\sum \begin{matrix} either\ m\in\sigma(\mathbb{N})and\ n\neq\sigma(\mathbb{N}) \\ or\ m\notin\sigma(\mathbb{N})and\ n\in\sigma(\mathbb{N}), \end{matrix} \\ 1 & if \qquad n=m\notin\cup_{\sigma\in\sum\sigma}(\mathbb{N}), \\ 0 & if \qquad n,m\notin\cup_{\sigma\in\sum\sigma}(\mathbb{N})and\ m\neq. \end{cases}$$

Then it is easily seen that

$$\forall m\in\mathbb{N},\ \sum_{n=1}^{\infty}\tilde{d}_{mn}=1,\quad \forall n\in\mathbb{N},\quad \sum_{n=1}^{\infty}\tilde{d}_{mn}=1.$$

According to Lemma .4, there exists a doubly stochastic operator$\widetilde{D}\in\mathcal{DS}$, such that $\widetilde{D}$ is represented by $\left(\tilde{d}_{mn}\right)_{m,n\in\mathbb{N}}$. To show that for each $\sigma\in, P_\sigma D=\widetilde{D}P_\sigma$ on c_0, it suffices to show their equality on the Schauder basis $\{e_n|n\in\mathbb{N}\}$ of c_0. For each$n\in\mathbb{N}$,

$$\widetilde{D}P_\sigma(e_n)=\widetilde{D}e_{\sigma(n)}=\sum_{m=1}^{\infty}\widetilde{D}e_{\sigma(n)}(m)e_m$$

$$=\sum_{m=1}^{\infty}d_{m\sigma(n)}e_m=\sum_{m\in\sigma(\mathbb{N})}De_n\left(\sigma^{-1}(m)\right)e_m$$

$$=\sum_{k=1}^{\infty}De_n(k)e_{\sigma(k)}=P_\sigma\left(\sum_{k=1}^{\infty}De_n(k)e_k\right)=P_\sigma D(e_n).$$

In the followingtheorem, we obtain the structure of linear preservers of majorization on c_0.

Theorem .16 For a bounded linear operator $T: c_0\to c_0$, [4]. the following conditions are equivalent.

(i) $T\in\mathcal{M}_{Pr}(c_0)$.

(ii) There exists $\alpha\in c_0$ and a mutually disjoint family of one-to-one maps $\sum=\{\sigma_i:\mathbb{N}\to\mathbb{N}|i\in I\}$, where $I=supp(\alpha)=\{i\in\mathbb{N}|\alpha_i:=\alpha(i)\neq 0\}$, for which $T=\sum_{i\in I}\alpha_i P\sigma_i$. Here the series is understood to converge in the operator norm topology of $B(c_0)$, the set of all bounded linear operators on c_0.

Proof. Let $T: c_0\to c_0$ be a non-zero bounded linear operator.

(i)⇒(ii) Since $T\neq 0$, there exists $n_0\in\mathbb{N}$ with $Te_{n_0}\neq 0$. Let $\alpha:=Te_{n_0}$ and $I:=\{i\in\mathbb{N}|Te_{n_0}(i)\neq 0\}$. For each $n\in\mathbb{N}$, since $Te_n\sim Te_{n_0}$, by Theorem .9,there exists a bijection $\theta_n:\mathbb{N}\to\mathbb{N}$ such that

$$Te_n=P_{\theta_n}\left(Te_{n_0}\right).$$

For $i\in I$, Let $\sigma_i:\mathbb{N}\to\mathbb{N}$ be defined, for each $n\in\mathbb{N}$, by $\sigma_i(n)=\theta_n(i)$. Then, by Lemma .14, each $\sigma_i$is a one-to-one map and σ_1,σ_2have disjoint ranges for distinct $i_1,i_2\in I$. It is easily seen that $\sum_{i\in I}\alpha_i P\sigma_i$ is a well-defined bounded linear operator on c_0. We show that $\sum_{i\in I}\alpha_i P\sigma_i$ converges in the operator norm topology to T. For each $f=\sum_{n=1}^{\infty} f(n)e_n \in c_n$ and $m\in\mathbb{N}$, we have

$$Tf-\sum_{i\in I,i\leq m}\alpha_i P\sigma_i(f)=\sum_{n=1}^{\infty} f(n)Te_n-\sum_{i\in I,i\leq m}\alpha_i\sum_{n=1}^{\infty} f(n)e_{\sigma_i(n)}$$

$$=\sum_{n=1}^{\infty} f(n)\,P_{\theta_n}\left(Te_{n_0}\right)-\sum_{i\in I,i\leq m}\sum_{n=1}\alpha_i f(n)e_{\sigma_i(n)}$$

$$=\sum_{n=1}^{\infty}\sum_{i\in I} f(n)P_{\theta_n}\left(Te_{n_0}(i)e_i\right)-\sum_{n=1}^{\infty}\sum_{i\in I}\alpha_i f(n)e_{\sigma_i(n)}$$

$$=\sum_{n=1}^{\infty}\sum_{i\in I}\alpha_i f(n)e_{\sigma_i(n)}-\sum_{n=1}^{\infty}\sum_{i\in I,i\leq m}\alpha_i f(n)e_{\sigma_i(n)}$$

$$=\sum_{n=1}^{\infty}\sum_{i\in I,i>m}\left(\alpha_i f(n)\right)e_{\sigma_i(n)},$$

and therefore, by mutually dis joint ness of the family $\sum$,

$$\left\|Tf-\sum_{i\in I,i>m}\alpha_i P\sigma_i(f)\right\|=\sup_{n\in\mathbb{N},i\in I,i>m}|\alpha_i f(n)|\leq\|f\|\sup_{i>m}\left|Te_{n_0}(i)\right|$$

Hence $\left\|T-\sum_{i\in I,i>m}\alpha_i P\sigma_i\right\|\leq sup_{i>m}\left|Te_{n_0}(i)\right|\to 0, as\ m\to\infty$.thus

$$T=\sum_{i\in I}\alpha_i P\sigma_i$$

(ii)⇒(i) For f and g in c_0, let $f=Dg$ for some $D\in\mathcal{DS}$. By Lemma .15, there exists $\widetilde{D}\in\mathcal{DS}$ such that $P_\sigma D=\widetilde{D}P_\sigma$, for each $\sigma\in\sum$. Therefore,

$$Tf=\sum_{i\in I}\alpha_i P\sigma_i(f)=\sum_{i\in I}\alpha_i P\sigma_i D(g)$$

$$=\sum_{i\in I}\alpha_i\widetilde{D}P\sigma_i(g)=\widetilde{D}\sum_{i\in I}\alpha_i P\sigma_i(g)$$

$$=\widetilde{D}(Tg)$$

i.e. $Tf\prec Tg$.

It is deduced from Theorem .16,that if a bounded linear map $T: c_0\to c_0$ is represented by an infinite matrix (t_{ij}), then T is a linear preserver if and only if the columns of this matrix are permutations of each other and in each row of it there exists at most one non-zero element. This structure is similar to that of linear preservers of majorization on ℓ^p spaces, with $1<p<\infty$, except in the fact that the columns of the latter belong to the space ℓ^p

while those of the former are in c_0. We now turn our attention towards the characterization of linear maps

$T \in \mathcal{M}_{\mathrm{Pr}}(\mathrm{c})$.. For each $T \in \mathcal{B}(\mathrm{c})$, let $T_0: \mathrm{c}_0 \to \mathrm{c}$ be the restriction of T to c_0. The following corollary is obtained directly from Theorem.14, part (i) and Theorem.16

Corollary .17 For a bounded linear operator $T: c \to c$, the following statements are equivalent.

(i)$T \in \mathcal{M}_{Pt}(c)$,

(ii) There exists a subset $I \subseteq \mathbb{N}$, a set of non-zero real numbers $\{\alpha_i | i \in I\}$ which, if infinite, belongs to $c_0(I)$, a mutually disjoint family of one-to-one maps

$\sum = \{\sigma_i: \mathbb{N} \to \mathbb{N} | i \in I\}$, and an element $h \in c$ with $h(n) = \lim h$, for each

$n \in \bigcup_{i \in I\sigma_i}(\mathbb{N})$, for which

$$\forall (f_1 + f_1) \in c, \quad T(f_1 + f_1) = \left(\sum_{i \in I} \alpha_i P\sigma_i\right)(f - (\lim(f_1 + f_1))e) + (\lim(f_1 + f_1))h.$$

Proof. (i)⇒(ii) Let $T \in \mathcal{M}_{\mathrm{Pt}}(\mathrm{c})$ and suppose $\{\sigma_i: \mathbb{N} \to \mathbb{N} | i \in I\}$ is as given in Corollary.18 . Let $h := Te$ which clearly belongs to c. Then Theorem.19, shows that $h(m) = \lim h$, for each $m \in \bigcup_{i \in I\sigma_i}(\mathbb{N})$.

Moreover, for each $(f_1 + f_1) \in c$,

$$\begin{aligned} T(f_1 + f_1) &= t((f_1 + f_1) - (\lim(f_1 + f_1))e) + T\big((\lim(f_1 + f_1))e\big) \\ &= T_0((f_1 + f_1) - (\lim(f_1 + f_1))e) + (\lim(f_1 + f_1))T(e) \\ &= \left(\sum_{i \in I} \alpha_i P\sigma_i\right)((f_1 + f_1) - (\lim(f_1 + f_1))e) + (\lim(f_1 + f_1))h. \end{aligned}$$

(ii)⇒(i) Let $(f_1 + f_1) \prec g$, i.e. $(f_1 + f_1) = Dg$ for some $D \in \mathcal{DS}$. By Lemma.15, there exists $\widetilde{D} \in \mathcal{DS}$ such that for all $i \in I, P\sigma_i D = \widetilde{D} P\sigma_i$. In addition, using the definition of $\widetilde{D}$ in the proof of this same lemma, it is easily seen that $\widetilde{D}(e_n) = e_n$, for each $n \notin \bigcup_{i \in I\sigma_i}(\mathbb{N})$. Therefore,

$$\begin{aligned} \widetilde{D}(h) = \widetilde{D}(h - (\lim h)e + (\lim h)e) &= \widetilde{D}\left(\sum_{n \in \mathbb{N}} (h(n) - \lim h)e_n\right) + (\lim h)\widetilde{D}e \\ &= \widetilde{D}\left(\sum_{n \notin \bigcup_{i \in I\sigma_i}(\mathbb{N})} (h(n) - \lim h)e_n + (\lim h)e\right) \\ &= \sum_{n \notin \bigcup_{i \in I\sigma_i}(\mathbb{N})} (h(n) - \lim h)e_n + (\lim h)e \\ &= \textstyle\sum_{n \in \mathbb{N}} (h(n) - \lim h)e_n + (\lim h)e = h. \text{ Thus,} \end{aligned}$$

$$T(f_1+f_1)=\left(\sum_{i\in I}\alpha_i P\sigma_i\right)((f_1+f_1)-(\lim(f_1+f_1))e)+(\lim(f_1+f_1))h$$

$$=\left(\sum_{i\in I}\alpha_i P\sigma_i\right)D(g-(\lim g)e)+(\lim g)h$$

$$=\left(\widetilde{D}\sum_{i\in I}\alpha_i P\sigma_i\,(g-(\lim g)e)+(\lim g)h\right)$$

$$=\widetilde{D}(Tg),$$

i.e. $T(f_1+f_1)\prec Tg$. Hence T is a linear preserver.

Corollary .18 If T is a preserver on c, then there exist $I\subseteq\mathbb{N}$, a set of non-zero real numbers $\{\alpha_i|i\in I\}$(which, if infinite, belongs to c_0(I)), and mutually disjoint family of one-to-one maps$\{\sigma_i:\mathbb{N}\to\mathbb{N}|i\in I\}$ such that $T_0=\sum_{i\in I}\alpha_i P\sigma_i$.

As the following example shows, there are bounded linear operators $T:c\to c$ whose restriction on c_0 acts as a linear preserver on this subspace, while T itself is not a preserver on c.

Theorem .19 For $T\in\mathcal{M}_{\mathrm{Pr}}$(c), let T_0 be represented in the form$\sum_{i\in I}\alpha_i P_{\sigma_i}$, as described in Corollary.18 . If $a=limTe$ then $Te(m)=a$, for each $m\in\bigcup_{i\in I}\sigma_i(\mathbb{N})$.

Proof. Suppose, on the contrary, that there exists $i_0\in I$ and $m_0\in\sigma i_0(\mathbb{N})$ such that $Te(m_0)\neq a$. Let$n_0:=\sigma-i_0(m_0)$. Then

$$Te_{n_0}=\sum_{i\in I}\alpha_i P_{\sigma_i}(e_{n_0})=\sum_{i\in I}\alpha_i e_{\sigma_i}(n_0). \tag{8}$$

Since $\{\sigma_i:\mathbb{N}\to\mathbb{N}|i\in I\}$ is a mutually disjoint family, it follows from (34) that

$$Te_{n_0}(m_0)=\sum_{i\in I}\alpha_i e_{\sigma_i(n_0)}(m_0).$$

Let $:=\alpha_{i0}$, and $d:=inf\{|\alpha-x||x\in\{\alpha_i|i\in I,\alpha_i\neq\alpha\}\cup\{0\}\}$. Then, since $\alpha\neq 0$ and the only limit point of $\{\alpha_i|i\in I\}$, if any, is $0,d$ is positive. If $N\in\mathbb{N}$ is chosen with $N>\frac{2\|Te\|}{d}$then

$$|\alpha N+Te(m_0)|\geq N|\alpha|-|Te(m_0)|>\frac{2\|Te\|}{d}|\alpha|-\|Te\|\geq\|Te\|. \tag{9}$$

Furthermore, since $T_0\in\mathcal{M}_{Pr}(c_0)$,, by Lemma.14 , there exists $n_1\in\mathbb{N}$, with $n_1>n_0$, such that such that:$\forall n\geq n_1,\forall m=1,\ldots,m_0,\ Te_n(m)=T_0e_n(m)=0$.

On the other hand, using the fact that $\mathrm{e}+\mathrm{Ne}_{\mathrm{n}_0}\sim\mathrm{e}+\mathrm{Ne}_{\mathrm{n}_1}$, we have

$Te + Ne_{n_0} \sim Te + Ne_{n_1}$. Thus, by Theorem.19 ,

$$Te(m_0) + \alpha N = (Te + NTe_{n_0})(m_0) \in \{(Te + NTe_{n_1})(m) | m \in \mathbb{N}\}.$$

By (35), the value $Te(m_0) + \alpha N$ does not belong to the image of Te. Hence

$$Te(m_0) + \alpha N \notin \{Te(1), \dots, Te(m_0)\} = \{(Te + NTe_{n_1})(1), \dots, (Te + NTe_{n_1})(m_0)\}.$$

Consequently, $Te(m_0) + \alpha N = (Te + NTe_{n_0})(m_0) = (Te + NTe_{n_1})(m_1)$ for some $m_1 > m_0$. Repeating a similar argument for m_1, n_1, in place of m_0, n_0, one can find two sequences $m_0 < m_1 < m_2 < \dots$ and $n_0 < n_1 < n_2 < \dots$ in $\mathbb{N}$, for which

$$\forall k \in \mathbb{N}, Te(m_0) + \alpha N = (Te + NTe_{nk})(m_k). \quad (10)$$

Since the sequence $(Te(m_k))_{k\in\mathbb{N}}$ converges, the sequence $(Te_{nk}(m_k))_{k\in\mathbb{N}}$ should also be convergent. On the other hand, since each $Te_{nk}(m_k)$ is member of $\{\alpha_i | i \in I\} \cup \{0\}$, we have $t := lim\, Te_{nk}(m_k) \in \overline{\{\alpha_i | i \in I\} \cup \{0\}} = \{\alpha_i | i \in I\} \cup \{0\}$. If $t = \alpha$ then, by (10), $Te(m_0) = lim\, Te = a$, which contradicts our assumption. Hence $t = \alpha$. Therefore, using (36) once more, we obtain the equality

$$Te(m_0) + \alpha N = \lim_{k\to\infty} (Te + NTe_{nk})(m_k) = \alpha Nt,$$

from which, by the fact that $|\alpha - t| \geq d$, it follows that

$$N = \frac{a - Te(m_0)}{\alpha - t} = \frac{|a - Te(m_0)|}{|\alpha - t|} \leq \frac{2\|Te\|}{d}.$$

This contradicts the choice of N. Our last theorem in this section gives the structure of a linear preserver on c.

Theorem .20 For a bounded linear operator $T: c \to c$, the following statements are equivalent.

(i) $T \in \mathcal{M}_{Pt}(c)$,

(ii) There exists a subset $I \subseteq \mathbb{N}$, a set of non-zero real numbers $\{\alpha_i | i \in I\}$ which, if infinite, belongs to $c_0(I)$, a mutually disjoint family of one-to-one maps $\sum = \{\sigma_i: \mathbb{N} \to \mathbb{N} | i \in I\}$, and an element $h \in c$ with $h(n) = \lim h$, for each $n \in \cup_{i\in I\sigma_i}(\mathbb{N})$, for which

$$\forall f \in c, \quad Tf = \left(\sum_{i\in I} \alpha_i P\sigma_i\right)(f - (limf)e) + (limf)h.$$

Proof. (i)⇒(ii) Let $T \in \mathcal{M}_{Pt}(c)$ and suppose $\{\sigma_i: \mathbb{N} \to \mathbb{N} | i \in I\}$ is as given in Corollary(18). Let $h := Te$ which clearly belongs to c. Then Theorem.19 , shows that $h(m) = \lim h$, for each $m \in \cup_{i\in I\sigma_i}(\mathbb{N})$.

Moreover, for each $f \in c$,

$$Tf = t(f - (\lim f)e) + T((\lim f)e) = T_0(f - (\lim f)e) + (\lim f)T(e)$$

$$= \left(\sum_{i\in I} \alpha_i P\sigma_i\right)(f - (\lim f)e) + (\lim f)h.$$

(ii)⇒(i) Let $f \prec g$, i.e. $f = Dg$ for some $D \in \mathcal{DS}$. By Lemma.15 , there exists $\widetilde{D} \in \mathcal{DS}$ such that for all $i \in I, P\sigma_i D = \widetilde{D}P\sigma_i$. In addition, using the definition of $\widetilde{D}$ in the proof of this same lemma, it is easily seen that $\widetilde{D}(e_n) = e_n$, for each
$n \notin \cup_{i\in I\sigma_i}(\mathbb{N})$. Therefore,

$$\widetilde{D}(h) = \widetilde{D}(h - (\lim h)e + (\lim h)e) = \widetilde{D}\left(\sum_{n\in\mathbb{N}} (h(n) - \lim h)e_n\right) + (\lim h)\widetilde{D}e$$

$$= \widetilde{D}\left(\sum_{n\notin\cup_{i\in I\sigma_i}(\mathbb{N})} (h(n) - \lim h)e_n + (\lim h)e\right)$$

$$= \sum_{n\notin\cup_{i\in I\sigma_i}(\mathbb{N})} (h(n) - \lim h)e_n + (\lim h)e$$

$= \sum_{n\in\mathbb{N}}(h(n) - \lim h)e_n + (\lim h)e = h$. Thus,

$$Tf = \left(\sum_{i\in I} \alpha_i P\sigma_i\right)(f - (\lim f)e) + (\lim f)h$$

$$= \left(\sum_{i\in I} \alpha_i P\sigma_i\right)D(g - (\lim g)e) + (\lim g)h$$

$$= \left(\widetilde{D}\sum_{i\in I} \alpha_i P\sigma_i (g - (\lim g)e) + (\lim g)h\right)$$

$$= \widetilde{D}(Tg),$$

i.e. $Tf \prec Tg$. Hence T is a linear preserver.

In this section, without being able to characterize the set of all linear preservers of majorization on ℓ^∞, we will introduce two classes of these operators, each presenting a feature which distinguishes these operators from those on c and c_0, as well as those on ℓ^p spaces, for $1 \le p < \infty$. We will also obtain some properties of operators in $\mathcal{M}_{\mathrm{Pr}}(\ell^\infty)$, the set of all linear preservers of majorization on ℓ^∞.In what follows, $\mathbb{N}^k$represents the set of all k −tuples of natural numbers, for some $k \in \mathbb{N}$.

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